Applications of Fractional Operators and Special Functions to Geometric Function Theory

Research Protocol

for Registration in Ph.D. Programme

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Certificate

This is to certify that this Research Protocol entitled **"Applications of Fractional Operators and Special Functions to Geometric Function Theory"** by **Monika Yadav,** submitted in fulfillment of the requirement for registration in Ph.D. program in DEPARTMENT OF MATHEMATICS under the Faculty of Science of JECRC University, Jaipur during the academic year 2025, is a bonafide record of work carried out under my guidance and supervision.

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1 INTRODUCTION

Definition 1.1. A *complex number* is denoted by the letter $\mathbf{z}$, is $\mathbf{z} = r + is$ which is made up of two real number r and s, Additionally, we depict $\mathbf{z}$ as an ordered pair of real numbers r and s denoted (r, s) to the plane R^2 where r is on $X-$axis and s on $Y-$axis.

The **complex-valued functions** $\Phi(\mathbf{z}) : \mathcal{C} \to \mathcal{C}$ depending on a single complex variable $\mathbf{z} = r + is$, which can be uniquely expressed as a complex combination

$$\Phi(\mathbf{z}) = \Phi(r + is) = h(r, s) + ik(r, s) \tag{1}$$

are two real functions, each depending on the two real variables r, s.

Definition 1.2. Analytic Function

At a point $\mathbf{z}_0$ the limit defines the derivative of $\Phi(\mathbf{z})$ within its domain

$$\Phi'(\mathbf{z}_0) = \lim_{\mathbf{z}\to\mathbf{z}_0} \frac{\Phi(\mathbf{z}) - \Phi(\mathbf{z}_0)}{\mathbf{z} - \mathbf{z}_0}. \tag{2}$$

(Ahlfors (1979))

Definition 1.3. If a complex function $\Phi(\mathbf{z})$ has power series expansion

$$\Phi(\mathbf{z}) = \chi_0 + \chi_1(\mathbf{z} - \mathbf{z}_0) + \chi_2(\mathbf{z} - \mathbf{z}_0)^2 + \chi_3(\mathbf{z} - \mathbf{z}_0)^3 + ... = \sum_{m=0}^{\infty} \chi_m(\mathbf{z} - \mathbf{z}_0)^m \tag{3}$$

it is referred to as an analytic function at point $\mathbf{z}_0 \in \mathcal{C}$. For all $\mathbf{z}$ converges to point $\mathbf{z}_0$. Any analytic function is infinitely differentiable which is also said to be holomorphic function, where $\chi_0 = \Phi(\mathbf{z}_0),\ \chi_1 = \Phi^{'}(\mathbf{z}_0),\ \chi_2 = \frac{1}{2}\Phi^{''}(\mathbf{z}_0)$ and in general,

$$\chi_m = \frac{\Phi^{(m)}(\mathbf{z})}{m!} \tag{4}$$

Therefore, a convergent power series is the usual *Taylor series*

$$\Phi(z) = \sum_{m=0}^{\infty} \frac{\Phi^{(m)}(\mathbf{z}_0)}{m!}(\mathbf{z} - \mathbf{z}_0)^m \tag{5}$$

for the function $\Phi(\mathbf{z})$ at the point $\mathbf{z}_0$.

Therefore, if a complex function is complex differentiable at each point in $\mathfrak{R}$ it is considered an analytic function in $\mathfrak{R}$.

The function of class $\Phi(\mathbf{z})$ which is denoted by A and is defined as

$$\Phi(\mathbf{z}) = \mathbf{z} + \sum_{m=2}^{\infty} \chi_m \mathbf{z}^m \tag{6}$$

in an open unit disk $\mathbf{U} = \{\mathbf{z} : |\mathbf{z}| < 1, \mathbf{z} \in \mathbf{C}\}$ where $\Phi(\mathbf{z})$ is normalized by
$\chi_0 = \Phi(0) = 0,\ \chi_1 = \Phi^{'}(0) = 1.$

Univalent Function and Multivalent Function

Definition 1.4. A function $\Phi : \mathcal{D} \to \mathcal{C}$ where $\Phi(\mathbf{z}_1) \neq \Phi(\mathbf{z}_2),\ for\, every\ \mathbf{z}_1, \mathbf{z}_2 \in \mathcal{D}$ with $\mathbf{z}_1 \neq \mathbf{z}_2$. in an analytic function is said to be Schilicht or one-to-one.

In some neighborhood of $\mathbf{z}_0$,a function is is considered locally univalent at point $\mathbf{z}_0 \in \mathcal{D}$. A subclass of A denoted by $\mathbf{S}$ includes all analytic univalent functions . A conformal mapping as a function defined if it is both analytic and univalent, because of its angle-preserving property (Duren (1983)).

The Koebe function, $\mathbf{K}(z)$, is a well-known example in class $\mathbf{S}$. defined by

$$\mathbf{k}(\mathbf{z}) = \mathbf{z} + \sum_{m=2}^{\infty} m\mathbf{z}^m = \frac{\mathbf{z}}{(1-\mathbf{z})^2}, \quad (\mathbf{z} \in \mathbb{U}) \tag{7}$$

which for several subclasses of univalent function , is an extremal function. With the exception of a slit along the negative real axis from $w = -\infty$ to $w = -\frac{1}{4}$ the domain that comparises the whole complex plane maps as one -one onto. (Pommerenke (1975) ,Graham and Kohr (2003)),Goodman (1983), and Duren (1983))

Definition 1.5. C_e represent the class of e- valent analytic function of the form

$$\Phi(\mathbf{z}) = \mathbf{z}^e + \sum_{k=m}^{\infty} \chi_{e+k} \mathbf{z}^{e+k}, (e, m \in N = \{1, 2, 3...\}) \tag{8}$$

where $|\mathbf{z}| < 1$ the function is said to be e-valent or multivalent if no value of the function taken more than e times such that $\Phi(z) = w_0$ has exactly e solutions.

where Φ is analytic and e-valent in $\mathbb{U}$.

Definition 1.6 (Subordination)**.** Assume that the analytic functions Φ and Ψ are defined in unit disk. If a Schwarz function $S(z)$ analytic in disk exists such that
$|S(z)| < 1, S(0) = 0, \mathbf{z} \in \mathbb{U}$, that

$$\Phi(z) = \Psi(S(z)), \quad (\mathbf{z} \in \mathbb{U}) \tag{9}$$

where $\mathbf{z} \in \mathbb{U}$. Then the function Φ is said to be *subordinate* to Ψ. We represent this subordination by $\Phi(z) \prec \Psi(z), \quad (\mathbf{z} \in \mathbb{U})$.

Definition 1.7 (Hadamard Product)**.** The Operator of Φ and Ψ is defined by

$$(\Phi * \Psi)(\mathbf{z}) = z + \sum_{k=m}^{\infty} \chi_k b_k \mathbf{z}^k = (\Psi * \Phi)(\mathbf{z}), \tag{10}$$

is said to be Hadamard product or convolution of Φ and Ψ.

Where $\Phi \in A$ is given by (6) and $\Psi \in A$ is given by

$$\Psi(\mathbf{z}) = \mathbf{z} + \sum_{k=m}^{\infty} b_k \mathbf{z}^k, (m \in N = \{1, 2, 3...\}). \tag{11}$$

Definition 1.8 (Majorization)**.** Φ is said to be majorized by Ψ, majorization relation is denoted by,

$$\Phi(z) \ll \Psi(z),$$

where $\Phi, \Psi \in A$ (MacGregor (1967))
if there exists a function $\phi \in A$, satisfying $|\phi(\mathbf{z})| \leq 1$, $\mathbf{z} \in \mathbb{U}$ such that

$$\Phi(\mathbf{z}) = \phi(\mathbf{z})\Psi(\mathbf{z}). \tag{12}$$

Definition 1.9. For $z \in C, |z| < 1$, then analytic function $q(z)$ defined as

$$q(\mathbf{z}) = 1 + q_1\mathbf{z} + q_2\mathbf{z}^2 + q_3\mathbf{z}^3 + ... + q_n\mathbf{z}^n ... = 1 + \sum_{m=1}^{\infty} q_m \mathbf{z}^m \tag{13}$$

where $R(q(z) > 0 \, and \, q(0) = 0$.
This class is denoted by Q i.e $q(z) \in Q$ iff $\mathfrak{q}(\mathbf{z}) \prec \frac{1+\mathbf{z}}{1-\mathbf{z}}$.[(Graham and Kohr (2003),)]

Classes of Convex and Starlike Functions

Definition 1.10. The class of Starlike function $\mathbf{S}^*$ was first studied by Alexander (1915). Φ is univalent in $\mathbb{U}$ w. r. to w_0 and if $\Phi(0) = 0$, then Φ maps $\mathbb{U}$ onto a starlike domain iff

$$\Re\left[\frac{\mathbf{z}\Phi'(\mathbf{z})}{\Phi(\mathbf{z})}\right] > 0, \quad \mathbf{z} \in \mathbb{U}.$$

Definition 1.11. For some real value β the class $\mathbf{S}^*(\beta)$ is introduced by Robertson (1936) as

$$\Re\left[\frac{\mathbf{z}\Phi'(\mathbf{z})}{\Phi(\mathbf{z})}\right] > \beta, \quad \mathbf{z} \in \mathbb{U}.$$

where $0 \leq \beta < 1,$. for univalent starlike function $\Phi(\mathbf{z})$.

If $\beta = 0$ then the class denoted as $\mathbf{S}^*(0)$ i.e $\mathbf{S}^*$, of univalent starlike function in disk.

Definition 1.12. A function Φ is said to be convex if it maps the open unit disk $\mathbb{U}$ conformally onto a convex region in the plane. The Set of all such convex function is represented by C.Let $\Phi \in A$ is univalent in $\mathbb{U}$. Then generates a convex image domain iff this condition holds:

$$\Re\left[1 + \frac{\mathbf{z}\Phi''(\mathbf{z})}{\Phi'(\mathbf{z})}\right] > 0, \quad \mathbf{z} \in \mathbb{U}.$$

In other words we can say that the any two point of $\mathbb{U}$, the straight line segment joining them remains completely inside $\mathbb{U}$.

This condition was first stated and Study by Small Shiel (1972).

Definition 1.13. The class $C(\beta)$ of convex function is intorduced by Robertson (1936) as

$$C(\beta) = \left\{\Phi \in A : \Re\left(1 + \frac{\mathbf{z}\Phi''(\mathbf{z})}{\Phi'(\mathbf{z})}\right) > \beta, \ \ z \in \mathbb{U}, \ 1 > \beta \geq 0 \ \right\}$$

when $\beta = 0$ then $C(0) = C$, Thus $C \subset \mathbf{S}^* \subset \mathbf{S}$.

Fractional derivative and integral operators

Definition 1.14. The *fractional derivative* of $\Phi(\mathbf{z})$ of order γ is given by Owa (1978) (Owa and Srivastava (1987)) and defined as:

$$\mathfrak{D}_{\mathbf{z}}^{\gamma}\Phi(\mathbf{z}) = \frac{1}{\Gamma(1-\gamma)}\frac{d}{d\mathbf{z}}\int_0^{\mathbf{z}} \frac{\Phi(t)}{(\mathbf{z}-l)^{\gamma}}dl, \qquad (0 \leq \gamma < 1), \tag{14}$$

where the multiplicity of $(\mathbf{z}-l)^{\gamma}$ is eliminated by assuming that $log(\mathbf{z}-l)$ is real for $(\mathbf{z}-l) > 0$. $\Phi(\mathbf{z})$ is an analytic function on the $\mathbf{z}$-plane including the origin, which is a simply connected region.

Definition 1.15. Following, the fractional derivative of order $m+\gamma$ can be experssed as

$$(\mathfrak{D}_{\mathbf{z}}^{m+\gamma})\Phi(\mathbf{z}) = \frac{d^m}{d\mathbf{z}^m}\mathfrak{D}_{\mathbf{z}}^{\gamma}\Phi(\mathbf{z}) \quad (0 \leq \gamma < 1; m \in N_0). \tag{15}$$

Srivastava (2001)

Definition 1.16. The formulation of a order γ defined by Owa (1978); Srivastava (2001)

$$\mathfrak{D}_{\mathbf{z}}^{-\lambda}\Phi(\mathbf{z}) = \frac{1}{\Gamma(\gamma)}.\int_0^{\mathbf{z}} \frac{\Phi(l)}{(\mathbf{z}-l)^{1-\gamma}}dl, \quad (\gamma < 0) \tag{16}$$

where $\Phi(\mathbf{z})$ is an analytic functions in complex $\mathbf{z}$-plane containing the origin $(\mathbf{z} = 0)$ which is a simply connected region, and the multiplicity of $(\mathbf{z}-l)^{\gamma-1}$ is removed by entailing $log(\mathbf{z}-l)$ to be real when $\mathbf{z} - l > 0$. By applying simple calculation, we get

$$\mathfrak{D}_{\mathbf{z}}^{-\delta}\Phi(\mathbf{z}) = \frac{1}{\Gamma(2+\delta)}\mathbf{z}^{1+\delta} + \sum_{k=2}^{\infty}\frac{\Gamma(k+1)}{\Gamma(k+1+\delta)}\chi_k\mathbf{z}^{k+\delta}, \tag{17}$$

where $\Phi(\mathbf{z})$ of the form (6) and $\Phi(\mathbf{z}) \in A$.

Definition 1.17. Ruscheweyh Stephan published his work proceeding his work "New Criteria for Univalent Functions, 1975" in which he work on a class $\mathbf{K}_m$ such that

$$\Re\frac{(\mathbf{z}^m\Phi)^{(m+1)}}{(\mathbf{z}^{m-1}\Phi)^{(m)}} > \frac{m+1}{2}, \mathbf{z} \in U, \ m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}, \tag{18}$$

and proved the result that Φ is univalent in $\mathbb{U}$.

It is clear that $\mathbf{K_1} = \mathbf{K}$, the class of convex functions, and $\mathbf{K_0} = S^*_{1/2}$, the class of starlike functions of order $1/2$. The result $\mathbf{K} \subset S^*_{1/2}$ provided by Strohacker in 1933 is added to this consequence. Gupta and Jain (1976).

Ruscheweyh (1975) proved the following important properties of the class $\mathbf{K_m}$.

Theorem 1.18. $\mathbf{K_{n+1}} \subset \mathbf{K_n}$ *holds for* $n \in N_0$.

Another important relation between $\mathbf{K}$ and $\mathbf{S}^*_{1/2}$ is $\Phi \in K \Leftrightarrow \mathbf{z}\sqrt{\Phi'(\mathbf{z})} \in \mathbf{S}^*_{1/2}$ is also extended to $\mathbf{K_m}$. Owa et al. (1986) proved the following properties of Ruscheweyh derivative $\mathfrak{D}^\beta\Phi(\mathbf{z})$ of a function f.

Theorem 1.19. *Let* $\Phi \in A$ *be in class* $\mathbf{S}^*$ *of starlike functions and gratify the form* $\mathfrak{D}^\alpha\Phi(\mathbf{z}) \neq 0$, $0 < |\mathbf{z}| < 1$ *for* $\beta \geq -1$. *Then* $\mathfrak{D}^\beta\Phi(\mathbf{z})$ *is also in class* $\mathbf{S}^*$.

Definition 1.20. Let the function $\Phi \in A$ for $m, n \in N_0 = N \cup \{0\}, \quad 0 \leq \alpha_1 \leq \alpha_2$, Eljamal and Darus (2011) define the operator

$$\mathfrak{D}^{n,m}_{\alpha_1,\alpha_2} f(z) = \mathbf{z} + \sum_{k=m+1}^{\infty} \left[\frac{1 + (\alpha_1 + \alpha_2)(k-1)}{1 + \alpha_2(k-1)}\right]^n C(m,k)\chi_k \mathbf{z}^k, \tag{19}$$

which is generalized Ruscheweyh derivatives operator.

Definition 1.21. The **Fractional Ruscheweyh-Goyal derivative** operator involving fractional differential operator established by Agarwal and Paliwal (2014)(Goyal and Goyal (2005), Parihar and Agarwal (2011)) is described as

$$\mathcal{J}^{\alpha,\eta}_{\mathfrak{p}}\Phi(\mathbf{z}) := \frac{\Gamma(\eta - \alpha + \nu + 2)}{\Gamma(\eta + 1)\Gamma(\nu + 2)}\mathbf{z}^{\mathfrak{p}} J^{\alpha,\eta,\nu}_{0,\mathbf{z}}(\mathbf{z}^{\eta-p}\Phi(\mathbf{z})), \tag{20}$$

$$= \mathbf{z}^{\mathfrak{p}} + \sum_{k=m+\mathfrak{p}}^{\infty} \chi_k \mathcal{B}^{\alpha,\mu}_{\mathfrak{p}}(k)\mathbf{z}^k,$$

where

$$\mathcal{B}^{\alpha,\mu}_{\mathfrak{p}}(k) := \frac{\Gamma(k - \mathfrak{p} + 1 + \eta)\Gamma(\nu + 2 + \eta - \alpha)\Gamma(k + \nu - \mathfrak{p} + 2)}{\Gamma(k - \mathfrak{p} + 1)\Gamma(k + \nu - p + 2 + \eta - \alpha)\Gamma(\nu + 2)\Gamma(1 + \eta)}.$$

In favor of univalent functions, the Fractional Ruscheweyh-Goyal derivative operator is proposed for $\mathfrak{p} = 1$ in equation (20), and for $\eta = \alpha$. The ordinary Ruscheweyh derivative $\mathfrak{D}^\alpha$ of order α replaces the fractional Ruscheweyh-Goyal derivative $\mathcal{J}^{\alpha,\eta}$.

Equation (20) can be able to inscribed in convolution structure as:

$$\mathfrak{J}^{\alpha,\eta}\Phi(\mathbf{z}) = \mathbf{z}\ {}_2F_1(\eta + 1, \nu + 2; \nu + 2 + \eta - \alpha; \mathbf{z}) * \Phi(\mathbf{z}). \tag{21}$$

Definition 1.22. Reimann-Liouville integral: The fractional integral $\mathfrak{D}^{-\beta}_{0,\zeta}$ with fractional order $\beta \in R^+$ of function $X(\zeta)$ is defined as

$$\mathfrak{D}_{0,\zeta}^{-\beta}X(\zeta) = (\mathbf{Y}_\beta * X(\zeta)) = \frac{1}{\Gamma\beta}\int_0^\zeta \frac{x(\tau)}{(\zeta-\tau)^{(1-\beta)}}d\tau \ \beta > 0, \tag{22}$$

is called Riemann -Liouville's integral of order β.

where $\mathbf{Y}_\beta$, the convolution kernel of order $\beta \in R^+$ for fractional intgral, is defined by

$$\mathbf{Y}_\beta(\zeta) = \frac{\zeta_+^{\beta-1}}{\Gamma(\beta)}. \tag{23}$$

where Γ is the well - known Euler Gamma function.

(Li et al. (2011))

Definition 1.23. Riemann- Liouville's derivative : The fractional derivative order β of function $X(\zeta)$ is given as

$${}_{RL}\mathfrak{D}_{0,\zeta}X(\zeta) = \frac{d^m}{d\zeta^m}\mathfrak{D}_{0,\zeta}^{-(m-\beta)}X(\zeta) = \frac{1}{\Gamma(m-\beta)}\frac{d^m}{d\zeta^m}\int_0^\zeta (\zeta-\tau)^{m-\beta-1}X(\tau)d\tau, \tag{24}$$

where $m-1 \leq \beta < m \in Z^+$. (Li et al. (2011))

Definition 1.24. The Caputo Fractional derivative: The Caputo Fractional derivative of order β of function $X(\zeta)$ is defined

$${}^c\mathfrak{D}_{0,\zeta}^{\beta}X(\zeta) = \mathfrak{D}_{0,\zeta}^{(m-\beta)}\frac{d^m}{d\zeta^m}X(\zeta) = \frac{1}{\Gamma(m-\beta)}\int_0^\zeta (\zeta-\tau)^{m-\beta-1}X^m(\tau)d\tau \tag{25}$$

in which $m-1 < \beta < m \in Z^+$.

From above definition , we can write this ${}_c\mathfrak{D}_{0,\zeta}^{\beta}X^n(\zeta) = {}_c\mathfrak{D}_{0,\zeta}^{\beta+n}X(\zeta)$.

(Li et al. (2011))

Definition 1.25. Hilfer fractional derivative : The Hilfer fractional derivative $\mathfrak{D}_{c^+}^{\alpha,\beta}(.)$ of function$f \in C^n(c,d)$ of order $n-1 < \alpha < n$ and type $0 \leq \beta \leq 1$ is defined by

$$D_{c^+}^{\alpha,\beta}\Phi(X) = \mathfrak{D}_{a^+}^{\gamma-\alpha}\frac{d}{dX}^n\mathfrak{D}_{c^+}^{(1-\beta)(n-\gamma)}\Phi(X) \tag{26}$$

and

$$D_{b^-}^{\alpha,\beta}\Phi(X) = \mathfrak{D}_{b^-}^{\gamma-\alpha}\{\frac{-d}{dX}\}^n\mathfrak{D}_{b^-}^{(1-\beta)(n-\alpha)}\Phi(X) \tag{27}$$

where $\mathfrak{D}_{c^+}^{\alpha}(.)$ and ${}_{RL}\mathfrak{D}_{c^+}^{\alpha}(.)$ are RL fractional integral and derivative.

(Sousa and Oliveira (2018))

Definition 1.26. Prabhakar Derivative :

Let the function $\Phi \in L'(c,d),\ 0 \leq c < t < d \leq \infty,$ and $\Phi * {}^{\gamma}_{\rho,m-\mu,\omega} \in \omega^{m,1}(c,d)$, where ${}^{\gamma}_{\rho,\mu,\omega}(t) = t^{\mu-1}\mathfrak{E}^{\gamma}_{\rho,\mu}(\omega t^{\rho})\ t \in R$ and $\mu, \omega\, \rho, \gamma \in C,\ R(\mu) > 0,\ R(\rho) > 0.$
The Prabhakar derivative is defined as

$$\mathfrak{D}^{\gamma}_{\rho,m-\mu,\omega,c^{+}}\Phi(t) = \frac{d^m}{dt^m}(\mathfrak{E}^{-\gamma}_{\rho,m-\mu,\omega,c^{+}}\Phi)(t), \tag{28}$$

where $\mathfrak{E}^{\gamma}_{\rho,\mu}(x) = \sum \frac{\Gamma(\gamma+k)}{\Gamma(\rho)\Gamma(\rho k+\mu)}\frac{x^k}{k!}$ and $\mu, \omega, \rho, \gamma \in C, R(\mu), R(\rho) > 0.$ (Polito and Tomovski (2018))

Definition 1.27. Prabhaker Integral : Let the function $\Phi \in L'(c,d), 0 \leq c < t < d \leq \infty.$ The Prabhaker integral is defined as

$$\mathfrak{E}^{\gamma}_{\rho,\mu,\omega,c^{+}}\Phi) = \int_a^t (t-y)^{\mu-1}\mathfrak{E}^{\gamma}_{\rho,\mu}[\omega(t-y)^{\rho}]\Phi(y)dy \tag{29}$$

where $\rho, \mu, \omega, \rho \in C$, and $R(\rho), R(\mu) > 0.$
(Polito and Tomovski (2018))

Definition 1.28. Atangana -Baleanu fractional integral: The Atangana Baleanu fractional integral of a cts defined as :

$${}^{AB}_{a}I^{\alpha}_{t}f(t) = \frac{1-\alpha}{AB(\alpha)}f(t) + \frac{\alpha}{AB(\alpha)\Gamma(\alpha)}\int_a^t (t-\tau)^{\alpha-1}f(\tau)d\tau (Meddahiet\ al.\ (2021)) \tag{30}$$

Special Function

Definition 1.29. Hypergeometirc Function: The hypergeometric function, one of the most important special functions in mathematical analysis, is crucial to many branches of pure and practical mathematics. ${}_2\mathcal{F}_1(a,b;c;z)$ is defined by the series

$${}_2\mathcal{F}_1(a,b;c;) = \sum_{m=0}^{\infty}\frac{(a)_m(b)m}{(c)_m}\frac{z^m}{m!}, \tag{31}$$

where $(a)_n$ represent the Pochhammer symbol extension of $(a)_n = a(a+1)(a+2)(a+3).....(a+n-1)$ ${}_2\mathcal{F}_1(a,b;c;)$satisfies a second-order linear differential equation and generalizes several classical functions, such as exponential, logarithmic, and trigonometric functions. In geometric function theory and fractional calculus, this function is utilized as a generating

function to generate new subclasses of analytic and univalent functions with specific geometric features. Because of its numerous uses, analytical flexibility, and strong connections to fractional operators, it is an essential tool for modern mathematical research.(Andrew et al. (1999))

Definition 1.30. Bessel Function: The Bessel function is crucial to fractional calculus and applied mathematics due to its close connections to many engineering and physical models. The Bessel function for the first kind of order p is given by,

$$\mathfrak{J}_{\mathfrak{n}} = \sum_{m=0}^{\infty} (\frac{(-1)^m}{m!(\Gamma(m+p+1)} (\frac{\mathbf{z}}{2})^{2m+p}. \tag{32}$$

In another form

$$\mathfrak{J}_n(x) = \frac{1}{\pi} \int_0^{\pi} cos(n\tau - xsin\tau) d\tau = \frac{1}{\pi} Re\{ \int_0^{\pi} (e^{i(n\tau - xsin\tau)}) d\tau. \tag{33}$$

(Andrew et al. (1999)) In fractional calculus, several integral and differential operators that help create generalized classes of univalent and analytic functions are defined using Bessel-type functions. Their structural properties and recurrence relations offer a useful framework for applying classical conclusions to fractional-order systems. Bessel functions are therefore a crucial mathematical tool for studying the fractional transformations and geometric properties of complex functions.

Definition 1.31. Mittag -Leffler Function: A family of special functions is formed by the Mittag-Leffler function. The Maclaurin series defines the one-parameter form that Gösta Mittag-Leffler presented in 1903:

$$\mathfrak{E}_{\alpha}(z) = \sum_{m=0}^{\infty} \frac{\mathbf{z}^m}{\Gamma(\alpha m + 1)}. \tag{34}$$

where $\Gamma(x)$ is the gamma function.
Similarly, the extended Mittag-Leffler function is another name for Wimann's 1905 invention of the two-parameter Mittag-Leffler function,

$$\mathfrak{E}_{\alpha,\beta} = \sum_{m=0}^{\infty} \frac{\mathbf{z}^m}{\Gamma(\alpha m + \beta)}. \tag{35}$$

(Shukla and Prajapati (2007))

Definition 1.32. Fox Wright Function: This series represent a Fox wright function

$$ {}_p\Psi_q \left[\begin{array}{cccc} (u_1, U_1), & (u_2, U_2) & \cdots, (u_p, U_P) \\ (v_1, V_1), & (v_2, V_2) & \ldots, (v_q, V_P) \end{array} ; Z \right] = \sum_{m=0}^{\infty} \frac{\Gamma(Um+u)}{\Gamma(Vm+v)} \frac{\mathbf{z}^m}{m!}, \tag{36} $$

where Γ stands for Euler's gamma function and notation

$\Gamma(Um+u) = \prod_{j=1}^{p} \Gamma(U_j m + u_j)$

Similarly $\Gamma(Vm+v) = \prod_{k=1}^{q} \Gamma(V_k m + v_k)$

The generalized hypergeometric function naturally extends to the function ${}_p\Psi_q$. Fox (1927) and wright (1935) introduced it, and both authors examined its asymptotic behavior for **z** if $\Delta > -1$, where $\Delta = \sum_{j=1}^{q} V_j - \sum_{i=1}^{p} U_i$.

If $\Delta \geq -1$, this equation (35) has a non-zero radius of convergence.

(Karp and Prilepkina (2020))

Definition 1.33. Hurwitz and Reimann zeta function : The half series

$$ \zeta(x, s) = \sum_{n=0}^{\infty} \frac{1}{(n+x)^s}, \tag{37} $$

for $x > 0$ is called the Hurwitz zeta function. when x=1 is called the Reimann zeta function and it is denoted by $\zeta(s)$. we can write this $\sum_{n=1}^{\infty} = 1 + \int_{1^+}^{\infty} \frac{dx}{X^s}$.

(Andrew et al. (1999))

2 LITERATURE REVIEW

The geometric properties of the analytic functions are covenanted and crammed by a branch of complex analysis known as *Geometric Function Theory*. Geometric function theory, in particular, is a field of mathematics that is exemplified by fascinating relationships between geometry and analysis. Even though it dates back to the 19^{th} century, fresh advancements are always being made. Due to the significance of algebraic geometry and function theory on compact Riemann surfaces in the development of "finite-gap" results for non-linear integrable systems—a developing field of mathematics with numerous connections to mathematical physics—geometric function theory has seen a veteran renaissance in recent decades. In the 20th century, the geometric theory of complex variable functions became a separate subfield of complex analysis because of Alexander (1915), Bieberbach (1916), Koebe (1907). One traditional topic being discussed is geometric function theory. However, it continues to find new uses in a wide range of fields, including fluid dynamics, non-linear integrable systems theory of partial differential equations, and more modern mathematical physics. Compared to other areas of mathematics, geometric function theory is relatively new. In the 18th century, Euler established the first idea, which has its roots in function theory. The 19^{th} century saw the development of modern function theory.

Experts in the field of univalent and multivalent functions owe a debt of gratitude to Bernardi, who has dedicated significant time to compiling an extensive bibliography on the subject (see Bernardi (1966)). The first volume spans the years 1907 to 1965 and lists 1694 papers. The subsequent volume covers the period from 1966 to 1975 and includes 1563 papers. Both volumes feature a comprehensive index that categorizes subtopics within this field and identifies documents relevant to each subtopic. Consequently, it is straightforward for experts to determine the status of various problems up to the year 1975. Currently, there are numerous books and monographs devoted to the study of univalent functions. Among these, we recall the works of Ahlfors (1973), Pommerenke (1975), Goodman (1983), Hallenbeck and MacGregor (1984), Miller and Mocanu (2000), and Mocanu et al. (1999). *Geometric Function Theory* (GFT) is a specialized area of complex analysis that examines the geometric aspects of analytic functions. This branch of mathematics demonstrates the close ties that exist between analytic structure and geometric intuition. Although its origins can be traced back to the nineteenth century, the subject continues to evolve, enriched by modern developments. Specifically, recent advances

in the theory of ”finite-gap” solutions for nonlinear integrable systems—an area that has close connections to modern mathematical physics—have been greatly aided by the impact of algebraic geometry and function theory on compact Riemann surfaces. Due in large part to the pioneering work of Alexander (1915), Bieberbach (1916), and Koebe (1907), the geometric theory of complex variable functions became a distinct subfield of complex analysis during the twentieth century. One of the recurring themes in this field is geometric function theory, which keeps discovering new uses in a variety of fields, including fluid dynamics, nonlinear partial differential equations, and different issues in contemporary mathematical physics.When compared to other areas of mathematics, GFT has a relatively recent history. The earliest concepts date back to Euler in the eighteenth century, whose function theory research prepared the way for the development of modern function theory in the nineteenth century. By creating an extensive library on the subject, Bernardi significantly advanced the systematic study of univalent and multivalent functions (Bernardi). His first book, which spans the years 1907 to 1965, contains 1,694 papers; his second volume, which spans the years 1966 to 1975, contains 1,563 papers. The extensive indices in both works that organize papers by subtopic make it easier for researchers to track the evolution of specific concerns up until 1975. Since then, univalent function theory has been the focus of numerous books and monographs, including those by Ahlfors (1973), Pommerenke (1975), Goodman (1983), Hallenbeck and MacGregor (1984),The study of univalent functions continues to play a major role in geometric function theory. Its development was influenced by the Riemann mapping theorem, Koebe's 1907 work, Gronwall's 1914 demonstration of the Area Theorem, and Bieberbach's 1916 finding about the second coefficient of normalized univalent functions. Univalent function theory became a distinct and thriving field of study as a result of these turning points. The Romanian Mathematical School played a major role in the development of the geometric theory of univalent functions. Among its most prominent contributors were the Cluj locals Mocanu and Alugareanu. In 1931, Alugareanu established necessary and sufficient criteria for the univalence of functions within the open unit disk, laying the foundation for the Romanian school of thought. Mocanu developed the differential subordinations approach in conjunction with Miller, introduced the notion of -convexity, and established univalence conditions for non-analytic functions. More recently, Mocanu's work on differential superordinations has deepened and broadened the theory. The traditional ideas of differentiation and integration from integer orders are extended to arbitrary real or complex (fractional) orders in the quickly developing discipline of fractional calculus.

Fractional calculus offers a smooth and seamless transition between the operations of differentiation and integration by extending the concept of n-fold differentiation or integration to an order R or C. This expanded framework makes it possible to study dynamic systems with memory and genetic characteristics, which are frequently seen in biological, engineering, and physical processes but are not sufficiently represented by traditional integer-order calculus. As a result, fractional calculus provides a solid mathematical basis for characterizing systems whose current behavior depends on both their instantaneous state and their complete past.

Fractional calculus originated with a letter from Gottfried Wilhelm Leibniz in 1695 asking what a derivative of order $\frac{1}{2}$ meant.Next. Later, mathematicians such as Liouville (1834), Riemann (1892), and Caputo (1967) made significant contributions to the theoretical development of fractional integrals and derivatives. Liouville proposed the concept of fractional integration for periodic functions, whereas Riemann created the Riemann–Liouville integral, which formed the basis for the conventional definition of fractional operators. The Caputo derivative was initially established in the 20th century and offers a more usable formulation for initial-value problems in engineering because it allows the addition of physically interpretable initial conditionsKilbas and Trujillo (2006)Oldham and Spanier (1974). In the twentieth century, several academics developed these ideas and formalized the theoretical framework of fractional differential equations (FDEs). Oldham and Spanier released The Fractional Calculus: Theory and Applications of Differentiation and Integration to Arbitrary Order, the first comprehensive book on the topic, in 1974. This made the topic more well-known among applied mathematicians and engineers. Later monographs, such as those by Samko, Kilbas, and Marichev (1993), Podlubny (1999), and Kilbas and Trujillo (2006), have systematically investigated the Riemann–Liouville, Caputo, Hadamard, Grunwald–Letnikov, and Atangana–Baleanu derivatives. A range of physical processes and boundary Kilbas and Trujillo (2006) Atangana2016. Recent developments in fractional calculus have brought novel operators with non-singular and non-local kernels, such as the Caputo–Fabrizio and Atangana–Baleanu derivatives. These operators provide smoother physical interpretations and steer clear of kernel singularities Caputo2015. Complex memory and hereditary real-world phenomena, including anomalous relaxation processes, heat conduction in fractal media, and epidemiological models, have been modeled using these operators Atangana2017,Abdon2018. Furthermore, fractional calculus has been included into numerical methods, machine learning frameworks, and optimization problems to enable accurate and efficient simulations of dynamic systems Diethelm2010,Li2015. The subclasses

of analytic functions associated with fractional operators have been explored by Agarwal et al. (2015); Agarwal and Paliwal (2014, 2015, 2020); Agarwal et al. (2017, 2019, 2020); Parihar et al. (2015).

Special functions are recognized as powerful analytical tools in mathematics, particularly for solving differential equations and characterizing complex phenomena. Among the most important are the Fox-Wright, Bessel, Mittag-Leffler, and hypergeometric functions. Andrew et al. (1999) discusses the hypergeometric function ${}_2F_1(a, b; c; z)$, which is frequently involved in boundary value and transformation problems. Numerous other transcendental forms are included in this unifying function. The Bessel function $J_n(x)$, which is also covered in Andrew et al. (1999), is crucial for physics problems involving wave propagation and cylindrical symmetry. An generalization of the exponential function, the Mittag–Leffler function was created by Gösta Mittag-Leffler and is currently a fundamental kernel in fractional calculus, particularly for fractional differential and integral operators Shukla and Prajapati (2007). The Fox–Wright function ${}_p\Psi_q$ generalizes the hypergeometric function and provides flexible convergence behavior that is useful in fractional and integral transforms Karp and Prilepkina (2020). It was proposed by Fox Fox (1927) and Wright wright (1935). Deep connections exist between these functions: the Mittag–Leffler and Fox–Wright functions naturally generalize the Bessel and zeta functions, which arise as limiting examples of hypergeometric functions. Their applications in geometric function theory are demonstrated by recent studies, where they aid in the definition of novel subclasses of analytic and univalent functions employing fractional operators. Therefore, special functions continue to play a crucial role in bridging the gap between classical analysis and contemporary mathematical advancements in geometric function theory and fractional calculus.

3 JUSTIFICATION FOR SELECTION OF RESEARCH AREA/TOPIC

Many authors have introduced new classes and subclasses of analytic functions; However, there is still a need to further develop these classes and investigate their properties in more depth. Furthermore, since the proposed study is dynamic in nature, continuous review of existing literature is necessary to ensure its relevance and progress. The importance of this research is further emphasized by its applications in various fields of science and engineering, which require the use of advanced mathematical tools to solve increasing challenges. Therefore, there is a constant need to develop new mathematical tools by integrating concepts from different branches of mathematics.

4 RESEARCH OBJECTIVES

1. To use fractional differential and integral operators to create and introduce new subclasses in the open unit disk of univalent or analytic functions.

2. To determine inclusion relations, distortion growth and growth theorems, and coefficient estimates for the recently established or newly introduced classes of analytic functions.

3. Applied fractional operators to explore results in geometric function theory, expanding the scope of classical results.

4. To explore geometric characteristics, such as univalence, convexity, close-to-convexity, starlikeness and subordination of functions defined by fractional operators.

5. To apply geometric function theory in statistic.

5 RESEARCH METHODOLOGY

Proposed methods to be used in research are follows as:

1. Special features included: Integrate special functions to generalize fractional operators and establish connections between geometric function theory and applied special functions.

2. Analytical derivation: Use mathematical tools such as subordination, convolution (Hadamard product) and differential inequalities to obtain key results, including coefficient bounds, distortion theorems, and inclusion relations.

3. Confirmation and Comparison: Compare newly obtained results with classical findings to ensure their novelty, generalizability and mathematical significance.

4. Calculation support: Use symbolic computation software (such as Mathematica or Maple) for algebraic verification and graphical depiction of fractional parameter effects.

Mathematical software like Mathematica and Matlab are proposed to be used to visualize the results. Latex software is mainly used to write the whole research proposal and thesis.

6 EXPECTED OUTCOME / HYPOTHESIS

1. Creating new feature classes: New subclasses of analytic and univariate functions will be created and analyzed, enriching the existing knowledge in GFT.

2. Establishment of analytical results: Many theoretical results will be obtained for these new classes, including coefficient bounds, subordination relations, distortion theorems, and inclusion properties.

3. Integration of mathematical branches: The research will build a bridge between fractional calculus, special functions and geometric function theory, and provide a unified mathematical framework.

4. Potential for further applications: The results can contribute to applied areas within engineering and physics such as partial differential equations, signal processing ,image Processing and mathematical modelling.

7 WORK PLAN / TIME LINES

We anticipate completing the project within 30 months after the start date. The preliminary action plan is as follows:

Phase 1: Introduction and review of literature

In 0–12 months, we want to complete the introduction and literature review. Up until the thesis is written, a continuous literature review will be conducted to gather the data needed at different phases of the planned study.

Phase 2: Examination of geometric characteristics of analytic function classes

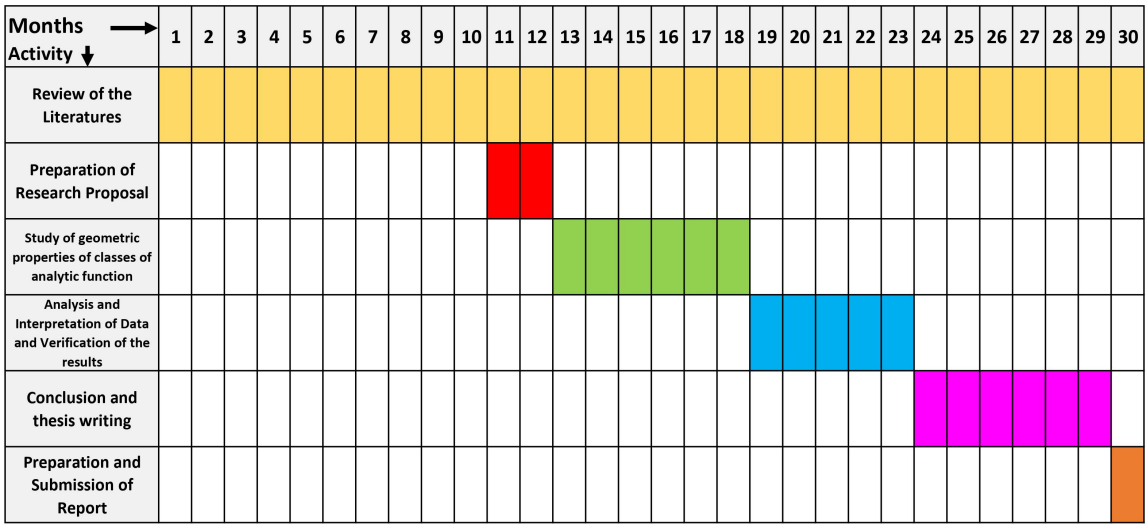
By reviewing the literature from phase 1, we will identify the gaps in this phase. These holes will be filled by introducing various classes of analytic functions utilizing incomplete functions, studying their geometric properties, and discovering some new conclusions.

We anticipate completing this phase in (6-24)months.

Phase 3: Conclusion and thesis writing

All of the work completed in earlier stages will be recorded in this step as a thesis.

This phase should be completed in 24 to 36 months.


Months
Activity
1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24 25 26 27 28 29 30
Review of the Literatures
Preparation of Research Proposal
Study of geometric properties of classes of analytic function
Analysis and Interpretation of Data and Verification of the results
Conclusion and thesis writing
Preparation and Submission of Report

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